Boxed plane partitions as an exactly solvable boson model

N M Bogoliubov

St Petersburg Department of V A Steklov Mathematical Institute, 27, Fontanka,
St Petersburg 191023, Russia

Abstract
Plane partitions naturally appear in many problems of statistical physics and
quantum field theory, for instance, in the theory of faceted crystals and of
topological strings on Calabi–Yau threefolds. In this paper, a connection is
made between the exactly solvable model with the boson dynamical variables
and a problem of enumeration of boxed plane partitions–three-dimensional
Young diagrams placed into a box of a finite size. The scalar product
of the introduced model gives the MacMahon enumeration formula, while
the correlation functions of the model may be considered as the generating
functionals of the Young diagrams with the fixed heights of its certain columns.
The evaluation of the scalar product and the correlation functions is based on
the Yang–Baxter algebra. The analytical answers are obtained in terms of
determinants and they can also be expressed through the Schur functions.

PACS numbers: 05.50.+q, 02.30.Ik

1. Introduction

The theory of plane partitions is a classical chapter in combinatorics [1]. Statistics of plane
partitions with respect to natural probabilistic measures was studied in [2, 3]. Locally, plane
partitions are equivalent to a tiling of a plane by rhombi (or lozenges). Among many results
obtained in this direction we should mention the paper [4] where the phenomenon of the 'arctic
circle' was described for boxed plane partitions. In the papers [5–7], the variational principle
applicable to a variety of such problems was developed. Correlation functions for random
plane partitions were studied in [7, 8].

Plane partitions naturally appear in many problems of statistical physics, for instance, in
the theory of faceted crystals [9] and of direct percolation [10]. Quite recently, it was argued
that there was a connection between topological strings on Calabi–Yau threefolds and crystal
melting [11].

In this paper, we demonstrate the connection of a certain integrable boson-type model and boxed plane partitions—the plane partitions placed into a box of a finite size. The natural dynamical variables in this boson model are the q-bosons. The algebra of q-bosons [12] appears naturally within the quantum algebra formalism [13].

Boxed plane partitions are related to the special case of q-bosons, namely to a limit when the deformation parameter q tends to zero which corresponds to an infinite value of coupling constant of the q-boson model. In this special limit q-bosons are known as exponential phase operators of quantum nonlinear optics [14]. Note that this is also the crystal limit for quantum groups [15, 16].

Our analysis is based on the quantum inverse scattering method (QISM) [17] and on the algebraic approach to the calculation of the correlation functions developed within this method [18]. We shall show that the scalar product of the state vectors of the phase model [19–21] is related to MacMahon enumeration formula for boxed plane partitions [22]. The systematic application of the QISM allows us to calculate different correlation functions appearing in the theory of the boxed plane partitions. There is a well-established connection between the theory of plane partitions and the theory of random processes. From that point of view the scalar product of a phase model is a generator of a point fermion-like random field [8, 9].

This paper is organized as follows. In section 2, we give some basic definitions of q-bosons. The QISM is reviewed in section 3 and the integrable q-boson model is introduced. In section 4, the scalar products of the state vectors of the introduced model are investigated and their connection with the enumeration problem for the boxed plane partitions is established. The expression of the state vectors in terms of symmetric Schur functions is given in section 5 and the certain correlation functions are calculated in section 6. In the last section, we discuss the relation of the boxed plane partitions with the q-difference Toda equation.

2. q-bosons

The q-boson algebra is defined by three independent operators B, $B^\dagger$ and N satisfying commutation relations

$$[B, B^\dagger] = q^{2N}, \qquad [N, B^\dagger] = B^\dagger, \qquad [N, B] = -B, \tag{1}$$

and q, a c-number, is taken to be $q = \mathrm{e}^{-\gamma}$. We consider real $\gamma > 0$. The q-boson algebra (1) has the representation in the Fock space formed from the q-boson normalized states $|n\rangle$,

$$B^\dagger |n\rangle = [n+1]^{\frac{1}{2}} |n+1\rangle, \qquad B|n\rangle = [n]^{\frac{1}{2}} |n-1\rangle, \tag{2}$$

where the 'box' is

$$[n] = \frac{1 - q^{2n}}{1 - q^2}. \tag{3}$$

The integer numbers $n > 0$ are called occupation numbers or the number of particles in a state $|n\rangle$:

$$N|n\rangle = n|n\rangle. \tag{4}$$

If $q = 1$ ($\gamma = 0$), the q-bosons become ordinary bosons,

$$B \rightarrow b, \qquad B^\dagger \rightarrow b^\dagger, \qquad N = b^\dagger b, \qquad [b, b^\dagger] = 1. \tag{5}$$

In the limit $q \rightarrow 0$ ($\gamma \rightarrow \infty$) the operators B, $B^\dagger$ transform into the operators ϕ, $\phi^\dagger$ defined by the commutation relations

$$[N, \phi] = -\phi, \qquad [N, \phi^\dagger] = \phi^\dagger, \qquad [\phi, \phi^\dagger] = \pi \tag{6}$$

in which π is the vacuum projector $\pi = |0\rangle\langle 0|$. The Fock states $|n\rangle$ can be created from the vacuum state $|0\rangle$ by operating by the phase operators,

$$|n\rangle = (\phi^\dagger)^n |0\rangle, \qquad N|n\rangle = n|n\rangle, \tag{7}$$

and

$$\phi^\dagger |n\rangle = |n + 1\rangle, \qquad \phi|n\rangle = |n - 1\rangle, \qquad \phi|0\rangle = 0. \tag{8}$$

One may verify that ϕ and $\phi^\dagger$ can be expressed in terms of the Fock states, $|n\rangle$, as

$$\phi = \sum_{n=0}^{\infty} |n\rangle\langle n + 1|, \qquad \phi^\dagger = \sum_{n=0}^{\infty} |n + 1\rangle\langle n|.$$

The introduced operator ϕ is 'one-sided unitary' or an isometric; although

$$\phi\phi^\dagger = 1,$$

one has

$$\phi^\dagger\phi = 1 - |0\rangle\langle 0|.$$

The operators (6) may be expressed in terms of ordinary bosons (5):

$$\phi = (b^\dagger b + 1)^{-\frac{1}{2}} b, \qquad \phi^\dagger = b^\dagger (b^\dagger b + 1)^{-\frac{1}{2}}.$$

3. Integrable phase model

The phase model is a special limit of the integrable q-boson model [19, 23]. It is defined by the L-operator [20, 21]:

$$L_n(u) \equiv \begin{pmatrix} a_n(u) & b_n(u) \\ c_n(u) & d_n(u) \end{pmatrix} = \begin{pmatrix} u^{-1} & \phi_n^\dagger \\ \phi_n & u \end{pmatrix}, \tag{9}$$

where parameter $u \in \mathcal{C}$, and $\phi_n, \phi_n^\dagger$ are the operators (6) satisfying commutation relations

$$[N_i, \phi_j] = -\phi_i \delta_{ij}, \qquad [N_i, \phi_j^\dagger] = \phi_i^\dagger \delta_{ij}, \qquad [\phi_i, \phi_j^\dagger] = \pi_i \delta_{ij}. \tag{10}$$

On the local Fock vectors

$$\phi_j |0\rangle_j = 0,$$
$$\phi_j^\dagger |n_j\rangle_j = |n_j + 1\rangle_j, \qquad \phi_j |n_j\rangle_j = |n_j - 1\rangle_j, \tag{11}$$
$$N_j |n_j\rangle_j = n_j |n_j\rangle_j,$$

and these states are orthogonal: $_k\langle n_m | n_i\rangle_j = \delta_{im}\delta_{kj}$.

The operator-valued matrix (9) satisfies the intertwining relation

$$R(u, v)(L_n(u) \otimes L_n(v)) = (L_n(v) \otimes L_n(u))R(u, v) \tag{12}$$

in which $R(u, v)$ is the 4×4 matrix with the non-zero elements equal to

$$R_{11}(u, v) = R_{44}(u, v) = f(v, u),$$
$$R_{22}(u, v) = R_{33}(u, v) = g(v, u), \tag{13}$$
$$R_{23}(u, v) = 1,$$

with

$$f(v, u) = \frac{u^2}{u^2 - v^2}, \qquad g(v, u) = \frac{uv}{u^2 - v^2}. \tag{14}$$

The symbol $\otimes$ denotes the tensor product of matrices: $(A \otimes B)_{ij,kl} = A_{ij}B_{kl}$. The R-matrix (13) satisfies the Yang–Baxter equation

$$(I \otimes R(u, v))(R(u, w) \otimes I)(I \otimes R(v, w)) = (R(v, w) \otimes I)(I \otimes R(u, w))(R(u, v) \otimes I).$$
(15)

The monodromy matrix is introduced as

$$T(u) = L_M(u)L_{M-1}(u)\cdots L_0(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}.$$
(16)

Matrix elements of this matrix act in the Fock space spanned on the state vectors

$$|n\rangle = \prod_{j=0}^{M} |n_j\rangle_j = \prod_{j=0}^{M} \left(\phi_j^\dagger\right)^{n_j} |0\rangle, \qquad \sum n_j = n,$$
(17)

where

$$|0\rangle = \prod_{j=0}^{M} |0\rangle_j$$
(18)

is the vacuum vector (generating state).

The commutation relations of the matrix elements of the monodromy matrix are given by the R-matrix (13)

$$R(u, v)(T(u) \otimes T(v)) = (T(v) \otimes T(u))R(u, v).$$
(19)

The most important relations are

$$C(u)B(v) = g(u, v)\{A(u)D(v) - A(v)D(u)\},$$
(20)

$$C(u)A(v) = f(v, u)A(v)C(u) + g(u, v)A(u)C(v),$$
(21)

$$D(u)B(v) = f(v, u)B(v)D(u) + g(u, v)B(u)D(v),$$
(22)

$$[B(u), B(v)] = [C(u), C(v)] = 0.$$
(23)

The relation (19) means that the transfer matrix $\tau(u) = \operatorname{tr} T(u) = A(u) + D(u)$ is the generating function of the integrals of motion: $[\tau(u), \tau(v)] = 0$ for all $u, v \in \mathcal{C}$.

The L-operator (9) satisfies the relation

$$e^{\zeta N_n} L_n(u) e^{\frac{1}{2}\zeta \sigma^z} = e^{\frac{1}{2}\zeta \sigma^z} L_n(u) e^{\zeta N_n}, \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
(24)

where parameter $\zeta \in \mathcal{C}$, and N_n is the number operator (11). From this equation and the definition of the monodromy matrix (16) it follows that

$$e^{\zeta \hat{N}} T(u) e^{\frac{1}{2}\zeta \sigma^z} = e^{\frac{1}{2}\zeta \sigma^z} T(u) e^{\zeta \hat{N}},$$
(25)

where

$$\hat{N} = \sum_{n=0}^{M} N_n$$
(26)

is a total number operator, and σ^z is the Pauli matrix (24). Equation (25) is equivalent to the set of equations

$$\hat{N}B(u) = B(u)(\hat{N} + 1), \qquad \hat{N}C(u) = C(u)(\hat{N} - 1).$$
(27)

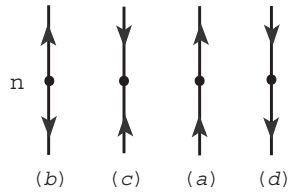It means that the operator $B(u)$ may be considered as a creation operator, while $C(u)$ as an annihilation one.

Figure 1. Vertex representation of the matrix elements of the L-operator.
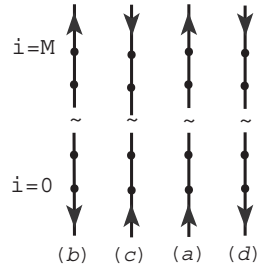

Figure 2. Matrix elements of the monodromy matrix $T(u)$.

The generating vector $|0\rangle$ (18) is annihilated by the operator $C(u)$ and is the eigenvector of operators $A(u)$ and $D(u)$:

$$C(u)|0\rangle = 0, \qquad A(u)|0\rangle = a(u)|0\rangle, \qquad D(u)|0\rangle = d(u)|0\rangle, \qquad (28)$$

with the eigenvalues $a(u) = u^{-(M+1)}$ and $d(u) = u^{M+1}$ respectively. The N-particle vectors are taken to be of the form

$$|\Psi_N(u_1, u_2, \ldots, u_N)\rangle = \prod_{j=1}^{N} B(u_j)|0\rangle, \qquad (29)$$

and from (27) it follows that

$$\hat{N}|\Psi_N(u_1, u_2, \ldots, u_N)\rangle = N|\Psi_N(u_1, u_2, \ldots, u_N)\rangle.$$

The vacuum state (18) is similar to the highest-weight vector in the theory of representations of Lie algebras. The state conjugated to (29) is

$$\langle\Psi_N(u_1, u_2, \ldots, u_N)| = \langle 0|\prod_{j=1}^{N} C(u_j). \qquad (30)$$

The dual vacuum $\langle 0| = |0\rangle^\dagger$, $\langle 0|0\rangle = 1$. It is easy to verify that $\langle 0|B(u) = 0$ and $\langle 0|A(u) = a(u)\langle 0|$, $\langle 0|D(u) = d(u)\langle 0|$.

One can visualize matrix elements of the L-operator as a vertex with the attached arrows (see figure 1). The matrix element $b_n(u) = \phi_n^\dagger$ corresponds to a vertex (b), $c_n(u) = \phi_n$ corresponds to a vertex (c), $a_n(u) = u^{-1}$ to a vertex (a) and $d_n(u) = u$ to a vertex (d) respectively.

Matrix elements of the monodromy matrix (16) are expressed then as sums over all possible configurations of arrows with different boundary conditions on a one-dimensional lattice with $M+1$ sites (see figure 2). Namely, the operator $B(u)$ corresponds to the boundary
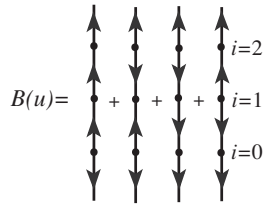
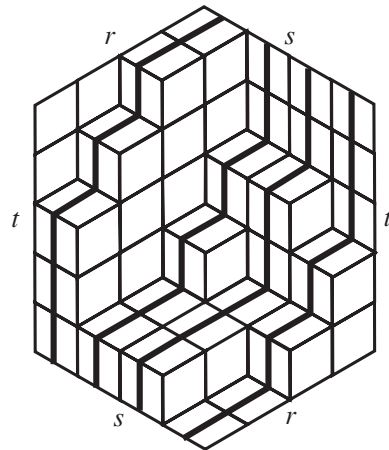**Figure 3.** Vertex representation of the operator $B(u)$.

Figure 4. A plane partition with gradient lines.

conditions when arrows on the top and bottom of the lattice point outwards (configuration (b)). Operator $C(u)$ corresponds to the boundary conditions when arrows on the top and bottom of the lattice point inwards (configuration (c)). Operators $A(u)$ and $D(u)$ correspond to the boundary conditions when arrows on the top and bottom of the lattice point up and down respectively (configurations (a) and (d)):

For example an operator $B(u)$ on the lattice of three sites ($M = 2$) is: $B(u) = \phi_0^\dagger u^{-2} + \phi_0^\dagger \phi_1 \phi_2^\dagger + u^2 \phi_2^\dagger + \phi_1^\dagger$, and may be represented in a form (see figure 3).

4. Scalar products and plane partitions

Recall that a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ is a non-increasing sequence of non-negative integers $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_N$ called the parts of λ. The sum of the parts of λ is denoted by $|\lambda|$.

A plane partition is an array π_{ij} of non-negative integers that are non-increasing as functions of both i and j ($i, j = 1, 2, \ldots$). The integers π_{ij} are called the parts of the plane partition, and $|\pi| = \sum \pi_{ij}$ is its volume. The plane partitions are often interpreted as stacks of cubes (three-dimensional Young diagrams). The height of a stack with coordinates ij is π_{ij}. If we have $i \leqslant r$, $j \leqslant s$ and $\pi_{ij} \leqslant t$ for all cubes of the plane partition, it is said that the plane partition is contained in a box with side lengths r, s, t. The symmetric plane partition is the plane partition for which $\pi_{ij} = \pi_{ji}$. In figure 4 the diagram corresponding to the boxed

plane partition $4 \times 4 \times 5$

$$
\pi = \begin{pmatrix} 5 & 3 & 3 & 2 \\ 5 & 2 & 1 & 1 \\ 4 & 1 & 1 & 0 \\ 3 & 1 & 1 & 0 \end{pmatrix} \tag{31}
$$

is represented.

A plane partition in a $r \times s \times t$ box is equivalent to a lozenge tiling of an (r, s, t)-semiregular hexagon. The term lozenge refers to a unit rhombus with angles of $\frac{\pi}{3}$ and $\frac{2\pi}{3}$. To each plane partition we can put into correspondence a q-weight $q^{|\pi|}$. Note that these q are not in any way related to the q-deformation parameter of section 2. The sum of q-weights of all plane partitions contained in a box $r \times s \times t$ is known as q-enumeration of plane partitions or the partition function of Young diagrams

$$
Z_q(r, s, t) = \sum_{p.p} q^{|\pi|} = \prod_{i=1}^{r} \prod_{j=1}^{s} \prod_{k=1}^{t} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}} = \prod_{j=1}^{r} \prod_{k=1}^{s} \frac{1 - q^{t+j+k-1}}{1 - q^{k+j-1}}. \tag{32}
$$

This formula is MacMahon generation function for the boxed plane partitions [22, 24]. In the $q \to 1$ limit this formula gives the number of the plane partitions containing in a box $r \times s \times t$.

Let us consider the scalar product of the state vectors (29) and (30):

$$
S(N, M|\{v\}, \{u\}) = \langle 0 | \prod_{j=1}^{N} C(v_j) \prod_{j=1}^{N} B(u_j) | 0 \rangle, \tag{33}
$$

where $\{u\}$ and $\{v\}$ are the sets of independent parameters. We shall show that after the parametrization

$$
u_j = q^{\frac{(j-1)}{2}}, \qquad v_j = q^{-\frac{j}{2}}, \tag{34}
$$

the scalar product (33) is the generating function (32) of the plane partitions containing in a box $N \times N \times M$:

$$
q^{\frac{N^2 M}{2}} \langle 0 | \prod_{j=-1}^{-N} C(q^{\frac{j}{2}}) \prod_{j=1}^{N} B(q^{\frac{j-1}{2}}) | 0 \rangle = Z_q(N, N, M). \tag{35}
$$

To make a connection of the scalar product (33) with the problem of enumeration of boxed plane partitions we shall use a graphic representation of its elements introduced in the previous section. Consider a two-dimensional square lattice with $2N \times (M + 1)$ sites. The first N columns of the lattice are associated with the operators $C(v_j)$ and the last N columns with operators $B(u_j)$ (see figures 1 and 2). The rows of the lattice are associated with the local Fock spaces, ith raw with the ith space respectively. Each horizontal edge of a lattice is labelled then with an occupation number n_j of a corresponding Fock vector $|n_j\rangle_i$. The scalar product (33) is equal to the sum over all allowed configurations on a square lattice with the arrows on the first N vertical rows pointing inwards, and on the last N ones pointing outwards; on the right and left boundaries all occupation numbers are zeros.

These configurations may be represented in terms of the lattice paths. The allowed configurations are then the number of possible non-crossing N paths starting from the N down left lattice edges $(-N; 0), (-N + 1; 0), \ldots, (-1; 0)$ and ending at the N top right ones $(1; M), (2; M), \ldots, (N; M)$. The mth path is running from $(-N + m - 1; 0)$ to $(m; M), 1 \leqslant m \leqslant N$. In the vertical direction paths follow the arrows, and only one path is allowed on a vertical lattice edge but any number of paths can share the horizontal ones. The number of paths sharing horizontal edges is equal to the corresponding occupation
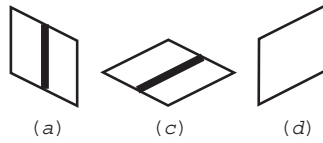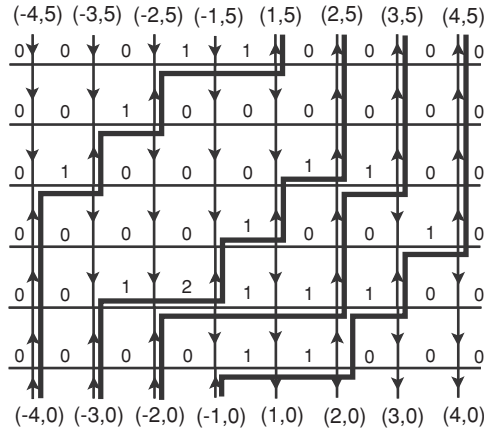
Figure 5. Three types of lozenges.

Figure 6. A typical configuration of admissible lattice paths.

number of the edge. The length of the paths is $(N + M)$. One of the possible configurations is represented in figure 6.

The cells of the lattice under the mth path may be considered as diagram of corresponding partition and may be thought of as the mth column in the array $(\pi_{i,j})$. The configuration of the paths in figure 6 corresponds to the plane partition in figure 4, and respectively to the array (31).

On the other hand we can associate the vertical and horizontal edges carrying paths with lozenges. Lozenge (a) in figure 5 corresponds to a vertical line of the path, while lozenge (c) to a horizontal one. Lattice edges without the paths correspond to a lozenge (d). The lozenge tiling is simply the projection of three-dimensional Young diagram with gradient lines. This establishes the mapping of the configurations generated by the scalar product (33) on the plane partitions.

Consider the scalar product (33). Due to commutation relations (23) it is a symmetric function of N variables v_j and also a symmetric function of N variables u_j. It is easy to verify that the number of operators $C(u)$ should be equal to the number of operators $B(u)$; otherwise the scalar product is equal to zero. The scalar product is evaluated by means of commutation relations (20)–(22). In the simplest case ($N = 1$) the scalar product is equal to

$$
\begin{aligned}
S(1, M | v, u) &= \langle 0 | C(v) B(u) | 0 \rangle \\
&= g(v, u)\{a(v)d(u) - a(u)d(v)\} \\
&= \left\{ \left(\frac{u}{v}\right)^{M+1} - \left(\frac{u}{v}\right)^{-M-1} \right\} \times \frac{1}{\frac{u}{v} - \left(\frac{u}{v}\right)^{-1}},
\end{aligned}
$$

where $g(u, v)$ is the element of R-matrix (14) and $a(u)$ and $d(u)$ are the eigenvalues of $A(u)$ and $D(u)$ (28).

For the arbitrary N one may obtain [18, 21]

$$S(N, M|\{v\}, \{u\}) = \langle 0| \prod_{j=1}^{N} C(v_j) \prod_{j=1}^{N} B(u_j)|0\rangle$$

$$= \left\{ \prod_{j>k} \left(\frac{v_k}{v_j} - \frac{v_j}{v_k} \right)^{-1} \prod_{l>m} \left(\frac{u_l}{u_m} - \frac{u_m}{u_l} \right)^{-1} \right\} \det H, \quad (36)$$

where the matrix elements of $N \times N$ matrix H are equal to

$$H_{jk} = \left\{ \left(\frac{u_k}{v_j} \right)^{M+N} - \left(\frac{u_k}{v_j} \right)^{-M-N} \right\} \times \frac{1}{\frac{u_k}{v_j} - \left(\frac{u_k}{v_j} \right)^{-1}}. \quad (37)$$

The parametrization (34) transforms the scalar product into

$$S(N, M|\{q\}) = (-1)^{\frac{N(N-1)}{2}} \left\{ \prod_{j>k} \left(q^{\frac{j-k}{2}} - q^{-\frac{j-k}{2}} \right)^{-2} \right\} \det \mathcal{H}, \quad (38)$$

where

$$\mathcal{H}_{jk} = \frac{s^{\frac{k+j-1}{2}} - s^{-\frac{k+j-1}{2}}}{q^{\frac{k+j-1}{2}} - q^{-\frac{k+j-1}{2}}}, \quad (39)$$

with $s = q^{M+N}$. The determinant of the matrix $\mathcal{H}$ was considered in [25] in connection with the alternating sign matrices enumeration problem and is equal to

$$\det \mathcal{H} = (-1)^{\frac{N(N-1)}{2}} \left\{ \prod_{j>k} \left(q^{\frac{j-k}{2}} - q^{-\frac{j-k}{2}} \right)^2 \right\} \prod_{1 \leqslant j,k \leqslant N} \frac{s^{\frac{1}{2}} q^{\frac{j-k}{2}} - s^{-\frac{1}{2}} q^{-\frac{j-k}{2}}}{q^{\frac{k+j-1}{2}} - q^{-\frac{k+j-1}{2}}}. \quad (40)$$

Therefore,

$$S(N, M|\{q\}) = q^{-\frac{N^2 M}{2}} \prod_{1 \leqslant j,k \leqslant N} \frac{1 - q^{N+M+j-k}}{1 - q^{k+j-1}}$$

$$= q^{-\frac{N^2 M}{2}} \prod_{1 \leqslant j,k \leqslant N} \frac{1 - q^{M-1+j+k}}{1 - q^{k+j-1}}. \quad (41)$$

Finally, we have obtained the equality (35) for the scalar product:

$$\langle 0| C\left(q^{-\frac{N}{2}} \right) \cdots C(q^{-1}) C\left(q^{-\frac{1}{2}} \right) B(1) B\left(q^{\frac{1}{2}} \right) \cdots B\left(q^{\frac{N-1}{2}} \right) |0\rangle = q^{-\frac{N^2 M}{2}} Z_q(N, N, M). \quad (42)$$

5. Coordinate form of state vectors and Schur functions

Using the explicit form of the operators $B(\lambda)$ we may rewrite the N-particle state vector (29) in the 'coordinate' form

$$|\Psi_N(u_1, u_2, \ldots, u_N)\rangle = \prod_{k=1}^{N} B(u_k)|0\rangle = \sum_{\substack{0 \leqslant n_0, n_1, \ldots, n_M \leqslant N \\ n_0 + n_1 + \cdots + n_M = N}} f_{\{n\}}(u_1, u_2, \ldots, u_N) \prod_{j=0}^{M} |n_j\rangle_j, \quad (43)$$

with the function f equal to

$$f_{\{n\}}(u_1, u_2, \ldots, u_N) = f_{(n_{j_1}, n_{j_2}, \ldots, n_{j_M})}(u_1, u_2, \ldots, u_N) = \sum_{\mathcal{B}} u_1^{t_1^d - t_1^a} u_2^{t_2^d - t_2^a} \cdots u_N^{t_N^d - t_N^a}. \quad (44)$$
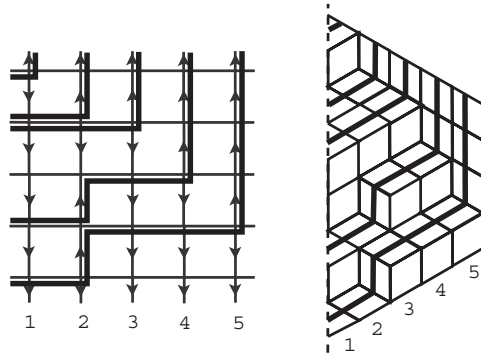
Figure 7. Lattice paths representation of a particular term of the 5-particle state vector, and a corresponding part of a plane partition.

Here the sum is taken over all admissible N paths with n_{j_1} paths starting from $(1; j_1)$, n_{j_2} from $(1; j_2)$ and n_{j_M} from $(1; j_M)$ respectively, $j_1 > j_2 > \cdots > j_M$; $n_{j_1} + \cdots + n_{j_M} = N, n_{j_k} \neq 0$. The power t_k^d is equal to the number of the $d(u)$ vertices in the kth vertical line of the grid, while t_k^a is equal to the number of $a(u)$ vertices. The graphic representation of the particular term of the state vector (43) with $N = 5$ is given in figure 7.

The following representation is valid,

$$f_{\{n\}}(u_1, u_2, \ldots, u_N) = (u_1 u_2 \ldots u_N)^{-M} S_{\{\lambda\}}(u_1^2, u_2^2, \ldots, u_N^2), \tag{45}$$

where $S_{\{\lambda\}}(u_1, u_2, \ldots, u_N)$ is the Schur function

$$S_{(\lambda_1, \ldots, \lambda_N)}(u_1, u_2, \ldots, u_N) = \frac{\det\left(u_j^{N-i+\lambda_i}\right)}{\prod_{1 \leqslant i < j \leqslant N}(u_i - u_j)}; \tag{46}$$

and there is a one-to-one correspondence between the occupation number configuration $\{n\}$ and partitions $\{\lambda\}$:

$$\{n\} = \left(n_{j_1}, n_{j_2}, \ldots, n_{j_M}\right) \quad \Longleftrightarrow \quad \{\lambda\} = (\lambda_1, \ldots, \lambda_N)$$

$$\lambda_k = j_1, \qquad 1 \leqslant k \leqslant n_{j_1}$$

$$\lambda_k = j_2, \qquad n_{j_1} + 1 \leqslant k \leqslant n_{j_2} + n_{j_1} \tag{47}$$

$$\vdots$$

$$\lambda_k = j_N, \qquad n_{j_1} + \cdots + n_{j_{M-1}} + 1 \leqslant k \leqslant n_{j_1} + \cdots + n_{j_M} = N.$$

The state conjugated to (43) is

$$\langle \Psi_N(u_1, u_2, \ldots, u_N)| = \langle 0| \prod_{k=1}^{N} C(u_k) = \sum_{\substack{0 \leqslant n_0, n_1, \ldots, n_M \leqslant N \\ n_0 + n_1 + \cdots + n_M = N}} f_{\{n\}}\left(u_1^{-1}, u_2^{-2}, \ldots, u_N^{-2}\right) \prod_{j=0}^{M} \langle n_j|_j, \tag{48}$$

with

$$f_{\{n\}}\left(u_1^{-1}, u_2^{-2}, \ldots, u_N^{-2}\right) = \sum_{\mathcal{C}} u_1^{t_1^d - t_1^a} u_2^{t_2^d - t_2^a} \cdots u_N^{t_N^d - t_N^a}, \tag{49}$$

where the sum is taken over all admissible N paths with n_{j_1} paths ending at $(-1; j_1)$, n_{j_2} ending at $(-1; j_2)$ and n_{j_M} ending at $(-1; j_M)$, $j_1 > j_2 > \cdots > j_M$; $n_{j_1} + \cdots + n_{j_M} = N, n_{j_k} \neq 0$.

Figure 8. Lattice path representation of a particular term of the conjugated state vector.

The lattice path representation of the particular term of the state vector (48) with $N = 5$ is given in figure 8.

Due to the orthogonality of the Fock states (17) the scalar product is equal to

$$S(N, M|\{v\}, \{u\}) = \langle \Psi_N(v_1, v_2, \ldots, v_N)|\Psi_N(u_1, u_2, \ldots, u_N)\rangle$$

$$= \sum_{\substack{0 \leqslant n_0, n_1, \ldots, n_M \leqslant N \\ n_0 + n_1 + \cdots + n_M = N}} f_{\{n\}}(v_1^{-1}, v_2^{-1}, \ldots, v_N^{-1}) f_{\{n\}}(u_1, u_2, \ldots, u_N)$$

$$= \sum_{\substack{0 \leqslant n_0, n_1, \ldots, n_M \leqslant N \\ n_0 + n_1 + \cdots + n_M = N}} \sum_{\mathcal{C}} v_1^{t_1^d - t_1^a} v_2^{t_2^d - t_2^a} \cdots v_N^{t_N^d - t_N^a} \sum_{\mathcal{B}} u_1^{t_1^d - t_1^a} u_2^{t_2^d - t_2^a} \cdots u_N^{t_N^d - t_N^a}. \quad (50)$$

Taking into account the representation (45) we can express the scalar product in terms of Schur functions

$$S(N, M|\{v\}, \{u\}) = \left(\prod_{j=1}^{N} \frac{v_j^M}{u_j^M} \right) \sum_{\lambda \subseteq \{M^N\}} S_{\{\lambda\}}(u_1^2, u_2^2, \ldots, u_N^2) S_{\{\lambda\}}(v_1^{-2}, v_2^{-2}, \ldots, v_N^{-2}), \quad (51)$$

where the sum is over all partitions, λ, into at most N parts each of which is less than or equal to M. Comparing this formula with equation (36) we obtain the following determinantal expression for the Schur functions,

$$\sum_{\lambda \subseteq \{M^N\}} S_{\{\lambda\}}(x_1^2, x_2^2, \ldots, x_N^2) S_{\{\lambda\}}(y_1^2, y_2^2, \ldots, y_N^2)$$

$$= \left(\prod_{j=1}^{N} x_j^M y_j^M \right) \left\{ \prod_{j>k} \frac{y_j y_k}{y_j^2 - y_k^2} \prod_{l>m} \frac{x_l x_m}{x_l^2 - x_m^2} \right\} \det H, \quad (52)$$

where the matrix elements of the matrix H (37) are equal to

$$H_{jk} = \{(x_k y_j)^{M+N} - (x_k y_j)^{-M-N}\} \times \frac{1}{x_k y_j - (x_k y_j)^{-1}}.$$

The volume $|\pi|$ of the plane partition π in a box $(N \times N \times M)$ may be expressed as

$$2|\pi| = N^2 M + \sum_{k=-1}^{-N} k(l_k^d - l_k^a) + \sum_{j=1}^{N}(j-1)(l_j^d - l_j^a), \quad (53)$$

where the first sum in this equality is over the columns going along the 's' side of hexagon while the second one is over the columns along the 'r' side, and $l_j^{a,d}$ is the number of lozenges of types

a, d (see figure 5) in the jth column of the hexagon. It may be checked that $l_j^d - l_j^a = t_j^d - t_j^a$. The substitution of parametrization (34) into equation (50) gives for the scalar product

$$S(N, M|\{q\}) = \left\langle \Psi_N\left(q^{-\frac{1}{2}}, q^{-1}, \ldots, q^{-\frac{N}{2}}\right) \middle| \Psi_N\left(1, q, \ldots, q^{\frac{N-1}{2}}\right)\right\rangle$$

$$= q^{-\frac{N^2 M}{2}} \sum_{p \cdot p} q^{|\pi|}.$$

Together with (41) this equation provides us with the proof of MacMahons enumeration formula for the boxed plane partitions within the frames of quantum inverse method.

Equation (51) gives the representation of the q-enumeration function in terms of Schur functions [26]:

$$Z_q(N, N, M) = \sum_{\lambda \subseteq \{M^N\}} S_{\{\lambda\}}(1, q, \ldots, q^{N-1}) S_{\{\lambda\}}(q, q^2, \ldots, q^N) = \sum_{p \cdot p} q^{|\pi|}.$$

The parts λ_k of the partition λ (47) may be considered as the coordinates of the particles (the coordinate λ_k corresponds to the kth particle). The state of the system is spanned by the orthonormal basis $|\lambda\rangle = |\lambda_1, \lambda_2, \ldots, \lambda_m\rangle : \langle \mu|\lambda\rangle = \delta_{\{\mu\},\{\lambda\}}$. The N-particle state vector (43) is then given by the equation

$$|\Psi_N(u_1, u_2, \ldots, u_N)\rangle = (u_1 u_2 \cdots u_N)^{-M} \sum_{\lambda \subseteq \{M^N\}} S_{\{\lambda\}}\left(u_1^2, u_2^2, \ldots, u_N^2\right)|\lambda\rangle, \tag{54}$$

and respectively

$$\prod_{j=1}^N \tilde{B}(u_j)|0\rangle = \sum_{\lambda \subseteq \{M^N\}} S_{\{\lambda\}}\left(u_1^2, u_2^2, \ldots, u_N^2\right)|\lambda\rangle, \tag{55}$$

where the sum in both formulae is taken over all partitions, λ, into at most N parts each of which is less than or equal to M. By the construction operators $\tilde{B}(u) \equiv u^{-M} B(u)$ and $\tilde{C}(u) \equiv u^M C(u)$ possess the following properties,

$$\tilde{B}(1)|\lambda\rangle = \sum_{\mu \supset \lambda} |\mu\rangle, \qquad \langle\lambda|\tilde{C}(1) = \sum_{\mu \supset \lambda} \langle\mu|, \tag{56}$$

and

$$\prod_{j=1}^N \tilde{B}(u_j)|\lambda\rangle = \sum_{\mu \supset \lambda} S_{\{\mu/\lambda\}}\left(u_1^2, u_2^2, \ldots, u_N^2\right)|\mu\rangle,$$

$$\langle\lambda| \prod_{j=1}^N \tilde{C}(u_j) = \sum_{\mu \supset \lambda} S_{\{\mu/\lambda\}}\left(u_1^{-2}, u_2^{-2}, \ldots, u_N^{-2}\right)\langle\mu|, \tag{57}$$

where $S_{\{\mu/\lambda\}}(x_1, x_2, \ldots, x_N)$ is a skew Schur function indexed by a pair of partitions μ and λ such that $\lambda \subset \mu$ [24]. From this point of view the operators $\tilde{B}(u)$ and $\tilde{C}(u)$ may be considered as the transition operators between the diagonals of plane partitions.

6. Correlation functions

Let us calculate the generating function of the plane partitions contained in a box $N \times N \times M$ provided that the height π_{1N} of the stack of the cubes is fixed and equal to m. To find this function we have to consider a scalar product on a lattice under the condition that the Nth lattice path enters the Nth vertical line of the grid at the mth row:

$$P_{1N}(m|\{v\}, \{u\}) = \langle 0| \prod_{j=1}^N C(v_j) \prod_{j=1}^{N-1} B(u_j) \phi_m^\dagger |0\rangle. \tag{58}$$

To calculate this scalar product we may use the following decomposition of the operators $B(u)$ and $C(u)$:

$$B(u)|0\rangle = u^{-M} \sum_{j=0}^{M} \phi_j^{\dagger} u^{2j} |0\rangle, \tag{59}$$

$$\langle 0|C(u) = u^M \langle 0| \sum_{j=0}^{M} \phi_j u^{-2j}. \tag{60}$$

The substitution of decomposition (59) into the scalar product (33) gives

$$S(N, M|\{v\}, \{u\}) = \sum_{m=0}^{M} (u_N)^{-M+2m} P_{1N}(m|\{v\}, \{u\}). \tag{61}$$

The determinant of the matrix H in (36) may be developed by the last column. The comparison of the obtained decomposition with (61) leads to the equality

$$\frac{P_{1N}(m|\{v\}, \{u\})}{S(N, M|\{v\}, \{u\})} = \frac{(-1)^{N-1}}{(u_N)^{N-1}} \left\{ \prod_{t<N} \frac{(u_N)^2 - (u_t)^2}{(u_t)^2} \right\} \frac{\det Q}{\det H}. \tag{62}$$

The entries of $N \times N$ matrix Q are given by

$$Q_{jN} = (v_j)^{M+N-1-2m}, \qquad Q_{jk} = H_{jk}, \quad k \neq N, \tag{63}$$

where H_{jk} are the matrix elements (37).

After the parametrization (34) we find that the probability of the height π_{1N} to be equal to m is given by the equation:

$$\begin{aligned} \langle m \rangle_{1N} &= \frac{P_{1N}(m|\{q\})}{S(N, M|\{q\})} \\ &= q^{-\frac{(N-1)^2}{2}} \left\{ \prod_{t<N} (1 - q^{N-t-2}) \right\} \frac{\det \mathcal{Q}}{\det \mathcal{H}}, \end{aligned} \tag{64}$$

where

$$\mathcal{Q}_{jN} = q^{-\frac{j(M+N-1-2m)}{2}}, \qquad \mathcal{Q}_{jk} = \mathcal{H}_{jk}, \qquad k \neq N, \tag{65}$$

and $\mathcal{H}_{jk}$ are the matrix elements (39). It is evident that this expectation value is the same for the height π_{N1} in the opposite corner of the diagram.

The correlation function of the heights of the columns π_{1N}, π_{N1} at the opposite sides of the Young diagram (see figure 9) may be obtained from the following scalar product:

$$P_{N1;1N}(n; m|\{v\}, \{u\}) = \langle 0|\phi_n \prod_{j=1}^{N-1} C(v_j) \prod_{j=1}^{N-1} B(u_j) \phi_m^{\dagger} |0\rangle. \tag{66}$$

The scalar products (61) and (66) may be expressed in terms of the skew Schur functions (57) as well.

The other function of interest is the projection of the Bethe wavefunction $\prod_{j=1}^{N} B(u_j)|0\rangle$ on the 'steady state' vector:

$$|P\rangle = \sum_{\substack{0 \leqslant n_0, n_1, \ldots, n_M \leqslant N \\ n_0 + n_1 + \cdots + n_M = N}} \prod_{j=0}^{M} |n_j\rangle_j. \tag{67}$$
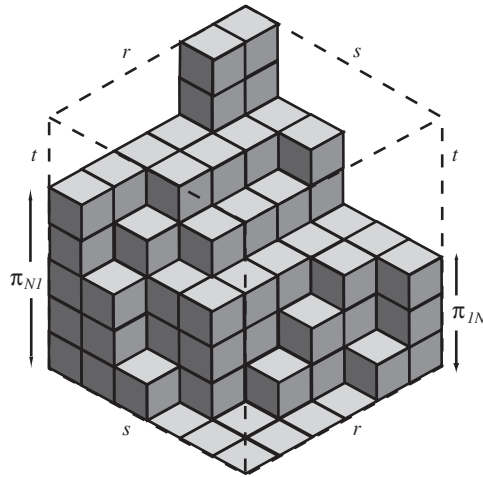
Figure 9. A Young diagram with fixed heights π of the stack of cubes.

From the representation (43) and the relation (45) we can express the considered projection in terms of the Schur functions

$$\langle P| \prod_{j=1}^{N} B(u_j)|0\rangle = (u_1 u_2 \cdots u_N)^{-M} \sum_{\lambda \subseteq \{M^N\}} S_{\{\lambda\}}\big(u_1^2, u_2^2, \ldots, u_N^2\big), \qquad (68)$$

where the sum is over all partitions, λ, into at most N parts each of which is less than or equal to M.

By the construction the considered correlation function is the q-enumeration function of the symmetric plane partitions, the plane partitions satisfying the condition $\pi_{ij} = \pi_{ji}$. From (44) and the relation $t_k^d - t_k^a = l_k^d - l_k^a$ it follows that

$$\langle P| \prod_{j=1}^{N} B(u_j)|0\rangle = \sum_{\mathcal{B}} u_1^{l_1^d - l_1^a} u_2^{l_2^d - l_2^a} \cdots u_N^{l_N^d - l_N^a}.$$

The volume of the symmetric plane partitions may be expressed as

$$2|\pi|_{\mathrm{sym}} = N^2 M + \sum_{j=1}^{N} (2j-1)\big(l_j^d - l_j^a\big), \qquad (69)$$

where the sum is over the columns going along the 'r' side of the hexagon. Then

$$\langle P| \prod_{j=1}^{N} B\big(q^{\frac{2j-1}{2}}\big)|0\rangle = q^{-\frac{N^2 M}{2}} \sum_{\mathrm{sym}} q^{|\pi|},$$

where the sum is taken over all symmetric plane partitions. On the other hand, from (68) the well-known result [22, 24] for the q-enumeration function of the symmetric plane partitions follows:

$$Z_q^{\mathrm{sym}}(N, N, M) = \sum_{\lambda \subseteq \{M^N\}} S_{\{\lambda\}}(q, q^3, \ldots, q^{2N-1}) = \sum_{\mathrm{sym}} q^{|\pi|}.$$

Till now we have considered plane partitions in a box with $r = s$. To study the general case when $r \neq s$ we have to consider the following scalar products:

$$S^A(N, L, M|\{v\}, \{u\}, \{u^A\}) = \langle 0| \prod_{j=1}^{N} C(v_j) \prod_{j=1}^{L} A(u_j^A) \prod_{j=1}^{N} B(u_j)|0\rangle, \quad (70)$$

$$S^D(N, L, M|\{v\}, \{u\}, \{u^D\}) = \langle 0| \prod_{j=1}^{N} C(v_j) \prod_{j=1}^{L} D(u_j^D) \prod_{j=1}^{N} B(u_j)|0\rangle. \quad (71)$$

Following the mapping introduced in the previous section it may be shown that these averages are the generating functions of plane partitions in a box $r \times s \times t$ with $r = N, s = N + L$, $t = M$ for (70), and $r = N + L, s = N, t = M$ for (71).

7. Boxed plane partitions and Toda lattice

The $N \times N$ matrix $\mathcal{H}$ (39) is a Hänkel matrix with the matrix elements

$$\mathcal{H}_{jk} = h(q^{j+k-1}), \quad (72)$$

where

$$h(q) = \frac{q^{\frac{M+N}{2}} - q^{-\frac{M+N}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}. \quad (73)$$

By the successive subtraction of columns the determinant of this matrix may be brought into the form

$$\det \mathcal{H} = \det h, \qquad h_{jk} = D_q^{j+k-2} h(q), \quad (74)$$

where D_q is the q-difference operator:

$$D_q f(z) = f(qz) - f(z). \quad (75)$$

Following the standard procedure [27, 28] it may be shown that the function $\tau(N, M; q) \equiv \det \mathcal{H}$ satisfies the equation

$$D_q^2 \ln \tau(N, M; q) = \frac{\tau(N + 1, M; q)\tau(N - 1, M; q)}{\tau^2(N, M; q)}, \quad (76)$$

which, after the substitution

$$\rho(N; q) = \ln \frac{\tau(N + 1, M; q)}{\tau(N, M; q)},$$

becomes the q-difference Toda equation

$$D_q^2 \rho(N; q) = e^{\rho(N+1;q)-\rho(N;q)} - e^{\rho(N;q)-\rho(N-1;q)}. \quad (77)$$

The role of time plays the weight q.

Acknowledgments

References

[1] Andrews G E 1998 *The Theory of Partitions* (Cambridge: Cambridge University Press)
[2] Vershik A 1996 Statistical mechanics of combinatorial partitions and their limit configurations *Funct. Anal. Appl.* **30** 90
[3] Vershik A and Kerov S 1977 Asymptotics of the Plancherel measure of the symmetric group and the limit form of Young tableaux *Sov. Math. Dokl.* **18** 527
[4] Cohn H, Larsen M and Propp J 1998 The shape of a typical boxed plane partition *New York J. Math.* **4** 137
[5] Destainville N, Mosseri R and Bailly F 1997 Configurational entropy of codimension-one tilings and directed membranes *J. Stat. Phys.* **87** 697
[6] Destainville N 1998 Entropy and boundary conditions in random rhombus tilings *J. Phys. A: Math. Gen.* **31** 6123
[7] Cohn H, Kenyon R and Propp J 2001 A variational principle for domino tilings *J. Am. Math. Soc.* **14** 297
[8] Okounkov A and Reshetikhin N 2003 Correlation function of Schur process with application to local geometry of a random 3-dimensional Young diagram *J. Am. Math. Soc.* **16** 58
[9] Ferrari P L and Spohn H 2003 Step functions for a faceted crystal *J. Stat. Phys.* **113** 1
[10] Rajesh R and Dhar D 1998 An exactly solvable anisotropic directed percolation model in three dimensions *Phys. Rev. Lett.* **81** 1646
[11] Okounkov A, Reshetikhin N and Vafa C 2003 Quantum Calabi–Yau and classical crystals *Preprint* hep-th/0309208
[12] Kulish P P and Damaskinsky E V 1990 On the q-oscillator and the quantum algebra $su_q(1, 1)$ *J. Phys. A: Math. Gen.* **23** L415
[13] Faddeev L D, Reshetikhin N Yu and Takhtajan L A 1990 Quantization of Lie groups and Lie algebras *Leningr. Math. J.* **1** 193
[14] Carruters P and Nieto M 1968 Phase and angle variables in quantum mechanics *Rev. Mod. Phys.* **40** 411
[15] Kashivara M 1990 Crystalizing the q-analogue of universal enveloping algebras *Commun. Math. Phys.* **133** 249
[16] Lusztig G 1993 Introduction to quantum groups *Progress in Mathematics* vol 110 (Basle: Birkhauser)
[17] Faddeev L D 1980 Quantum inverse scattering method *Sov. Sci. Rev. Math.* C **1** 107
[18] Korepin V E, Bogoliubov N M and Izergin A G 1993 *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge: Cambridge University Press)
[19] Bogoliubov N M, Bullough R K and Timonen J 1994 Critical behavior for correlated strongly coupled boson systems in 1 + 1 dimensions *Phys. Rev. Lett.* **25** 3933
[20] Bogoliubov N M and Nassar T 1997 On the spectrum of the non-Hermitian phase-difference model *Phys. Lett.* A **234** 345
[21] Bogoliubov N M, Izergin A G and Kitanine N A 1998 Correlation functions for a strongly correlated boson systems *Nucl. Phys.* B **516** 501
[22] Bressoud D M 1999 *Proofs and Confirmations. The Story of the Alternating Sign Matrix Conjecture* (Cambridge: Cambridge University Press)
[23] Kulish P P 1981 Quantum difference nonlinear Schrödinger equation *Lett. Math. Phys.* **5** 191
[24] Macdonald I G 1995 *Symmetric Functions and Hall Polynomials* (Oxford: Clarendon)
[25] Kuperberg G 1996 Another proof of the alternating-sign matrix conjecture *Int. Math. Res. Not.* **1996** 139
[26] Krattenthaller C 1995 The major counting of nonintersecting lattice paths and generating functions of tableaux *Mem. Am. Math. Soc.* **115** 552
[27] Izergin A G, Coker D A and Korepin V E 1992 Determinant formula for the six-vertex model *J. Phys. A: Math. Gen.* **25** 4315
[28] Izergin A G, Karjalainen E and Kitanine N A 2000 Integrable equations for the partition function of the six-vertex model *J. Math. Sci.* **100** 2141